

Determinantal Identities for Modular Schur Symmetric Functions

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Abstract

Modular symmetric functions are a new class of symmetric functions which depend both on a partition λ and an integer modulus $p > 2$. For p prime, these functions have representation theoretic significance as the irreducible characters of the general linear group $GL(n, K)$ where K is of characteristic p . In this paper we use classical algebraic techniques to prove determinantal identities that are modular analogues to the Jacobi–Trudi, dual Jacobi–Trudi, and Giambelli identities for the classical Schur functions.

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1 Introduction

Modular symmetric functions have been explored extensively by Doty and Walker [2] and Walker [7] in connection with determining simple p -modular polynomial characters of $GL(n, K)$ where K is algebraically closed and is of characteristic p . These modular symmetric functions, unlike the classical symmetric functions, depend on the integer modulus $p > 0$. This modulus produces a number of different effects. It causes the modular complete symmetric function to be a truncated version of its classical counterpart, but also requires the modular elementary symmetric function to have terms of higher degree than the classical elementary symmetric function. In this paper we prove determinantal identities for these functions. Specifically, we prove modular analogues of the Jacobi–Trudi, dual Jacobi–Trudi, Giambelli and skew Giambelli identities. As part of this, we introduce a skew version of the modular Schur function and also a ratio of alternants form.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are nonnegative integers and $\lambda_1 + \lambda_2 + \dots + \lambda_n = m$. Let $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n$. We then say λ is a partition of m (denoted $\lambda \vdash m$) with n parts. A partition can be represented by a (Ferrers) diagram which is a top and left justified arrangement of boxes such that the i th row contains λ_i boxes. The *conjugate* λ' of a partition λ is defined to be the partition whose diagram is the transpose of the diagram of λ . We can also define a *skew* partition. Given two partitions λ and μ we say $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all $i \geq 1$. Geometrically this means the diagram of λ contains the diagram of μ in its upper left hand corner. If we remove the diagram of μ from the upper left hand corner of the diagram of λ , then the resulting diagram is the diagram of the skew partition λ/μ . We say a diagram that is not of skew shape has standard shape. The conjugate of λ/μ is the skew partition λ'/μ' .

A partition λ can also be represented in so-called Frobenius notation as $\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r) = (\alpha | \beta)$, where the diagram of λ has r boxes on the main diagonal, and where $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i$. Note that α_i is the number of boxes in the i th row of the diagram of λ and to the right of the (i, i) box, and β_i is the number of boxes in the i th column of the diagram of λ and below the (i, i) box in the diagram of λ . Note that $(a | b)$ is the Frobenius notation for $\lambda = (a + 1, 1^b)$ (so that $(\alpha_i | \beta_i)$ represents the i th principal hook in the diagram), and that, if a or b is negative, we define $s_{(a | b)}(x) = 0$ if $a + b \neq 1$ and $s_{(a | b)}(x) = (-1)^b$ if $a + b = -1$.

2 Classical and Modular Symmetric Functions

We recall briefly definitions and results from classical symmetric function theory. Full details can be found in Macdonald [6].

The complete symmetric functions are defined by the generating function

$$H(t) = \sum_{d \geq 0} h_d(x) t^d = \prod_{i \geq 1} (1 - x_i t)^{-1}.$$

The elementary symmetric functions are defined by the generating function

$$E(t) = \sum_{d \geq 0} e_d(x) t^d = \prod_{i \geq 1} (1 + x_i t).$$

Note that $H(t)E(-t) = 1$, implying $\sum_{d=0}^n h_d(x)e_{n-d}(x) = 0$.

The classical Schur function can be defined in a number of different ways. We define the Schur function, $s_\lambda(x)$, as

$$s_\lambda(x) = \det(h_{\lambda_i - i + j}(x)). \quad (1)$$

The skew Schur function is defined as

$$s_{\lambda/\mu}(x) = \det(h_{\lambda_i - \mu_j - i + j}(x)). \quad (2)$$

Note that for $\mu = \emptyset$, (2) reduces to (1). A dual version of (2) is

$$s_{\lambda/\mu}(x) = \det(e_{\lambda'_i - \mu'_j - i + j}(x)). \quad (3)$$

If we restrict ourselves to a finite set of variables and to a standard shape partition, then the Schur function has a representation as a ratio of alternants, namely,

$$s_\lambda(x) = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n - j})}. \quad (4)$$

In Section 3 we prove a modular analogue of this result.

Consider now the modular symmetric functions.

For any integer $p \geq 2$, we define the *modular complete symmetric function* h'_d ($d \geq 0$) to be the symmetric function obtained by truncating the complete symmetric function h_d at the p th power of the variables. Formally, the generating function (Doty and Walker [2]) is given by

$$H'(t) = \sum_{d \geq 0} h'_d(x) t^d = \prod_{i \geq 1} \left(\frac{1 - x_i^p t^p}{1 - x_i t} \right).$$

In the particular case of $p = 2$, $h'_d = e_d$, the classical elementary symmetric function.

For any integer $p \geq 2$, we define the *modular elementary symmetric function* e'_d ($d \geq 0$) using the following generating function (Doty and Walker [2]):

$$E'(t) = \sum_{d \geq 0} e'_d(x) t^d = \prod_{i \geq 1} \left(\frac{1 + x_i t}{1 + (-1)^{p+1} x_i^p t^p} \right).$$

Note that $H'(t)E'(-t) = 1$, implying that

$$\sum_{d=0}^n (-1)^d e'_d(x) h'_{n-d}(x) = 0 \quad (5)$$

for all $n \geq 1$.

Various connections between the modular complete and elementary symmetric functions and their classical counterparts have been explored by Doty and Walker [2] and we refer the interested reader there.

Using the modular complete symmetric functions we can define modular Schur functions. Walker [7] has defined them for *standard shape* λ ; we extend this definition to skew shape:

$$s'_{\lambda/\mu}(x) = \det(h'_{\lambda_i - \mu_j - i + j}(x)). \quad (6)$$

In the next section we will see a ratio of alternants interpretation for the modular Schur function.

3 Determinantal Identities

As seen in (4), the classical Schur function has an interpretation as a ratio of alternants. In Theorem 3.1 we introduce a ratio of alternants equal to the modular Schur function for $p > n - l$. (This restriction is a limitation of the proof technique). The proof follows the techniques of Macdonald [6, p.25]. A key step in our proof involves splitting a sum. This step does not occur in the classical case because, although the upper limit of the sum is α_i , the sum is naturally limited to n since $e_r(x) = 0$ for $r > n$. This is not the case for modular elementary symmetric functions $e'_r(x)$ which can include terms of degree $> r$. Note, however, that, as in the classical case, the denominator term in Theorem 3.1 is the Vandermonde determinant.

Theorem 3.1 For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ and a modulus $p > n - 1$,

$$\det(h'_{\lambda_i - i + j}(x)) = \frac{\det \begin{pmatrix} (\sum_{j=n+1}^{\lambda_i + n - i} h'_{\lambda_i - i + j}(x)(-1)^{n-j} e'_{n-j}^{(k)}(x))_{1 \leq i \leq \lambda_i + n - p, 1 \leq k \leq n} \\ \dots\dots\dots \\ (\sum_{j=n+1}^{\lambda_i + n - i} h'_{\lambda_i - i + j}(x)(-1)^{n-j} e'_{n-j}^{(k)}(x) + x_k^{\lambda_i + n - i})_{\lambda_i + n - p < i \leq n, 1 \leq k \leq n} \end{pmatrix}}{\det(x_i^{n-j})}$$

Proof: Let $M' = ((-1)^{n-i} e'_{n-i}^{(k)}(x))_{1 \leq i, k \leq n}$ where $e'_r^{(k)}(x)$ denotes the modular elementary symmetric function of degree r in the variables $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$ (omitting x_k).

Let $H'_\alpha = (h'_{\alpha_i - n + j}(x))_{1 \leq i, j \leq n}$. Let

$$E'^{(k)}(t) = \sum_{r=0}^{n-1} e'_r^{(k)}(x) t^r = \prod_{i \neq k} \left(\frac{1 + x_i t}{1 + (-1)^{p+1} x_i^p t^p} \right).$$

Then

$$H'(t) E'^{(k)}(-t) = (1 - x_k t)^{-1} (1 - x_k^p t^p).$$

Equate the coefficient of t^{α_i} on either side to obtain

$$\sum_{j=1}^{\alpha_i} h'_{\alpha_i - n + j}(x) (-1)^{n-j} e'_{n-j}^{(k)}(x) = \begin{cases} x_k^{\alpha_i} & \text{if } \alpha_i < p \\ 0 & \text{else} \end{cases} \quad (7)$$

This implies

$$\sum_{j=1}^n h'_{\alpha_i - n + j}(x) (-1)^{n-j} e'_{n-j}^{(k)}(x) + \sum_{j=n+1}^{\alpha_i} h'_{\alpha_i - n + j}(x) (-1)^{n-j} e'_{n-j}^{(k)}(x) = \begin{cases} x_k^{\alpha_i} & \text{if } \alpha_i < p \\ 0 & \text{else} \end{cases}$$

$$A'_\alpha := H'_{\alpha_i} M' = \begin{bmatrix} (\sum_{j=n+1}^{\alpha_i} h'_{\alpha_i - n + j}(x) (-1)^{n-j} e'_{n-j}^{(k)}(x))_{1 \leq i \leq n \text{ such that } \alpha_i \geq p, 1 \leq k \leq n} \\ \dots\dots\dots \\ (\sum_{j=n+1}^{\alpha_i} h'_{\alpha_i - n + j}(x) (-1)^{n-j} e'_{n-j}^{(k)}(x) + x_k^{\alpha_i})_{1 \leq i \leq n \text{ such that } \alpha_i < p, 1 \leq k \leq n} \end{bmatrix} \quad (8)$$

If we take determinants of both sides of (8) we obtain

$$\det(H'_\alpha) \det(M') = \det(A'_\alpha).$$

Let $\delta = (n-1, n-2, \dots, 1, 0)$ and note that H'_δ is upper unitriangular so that $\det(H'_\delta) = 1$. If $\alpha_i = \delta_i = n-i$, the upper limit on the sum in (7) is $n-i$. If $n-1 \geq p$, then

$$A'_\delta = \begin{bmatrix} (0)_{1 \leq i \leq n-p, 1 \leq k \leq n} \\ \dots\dots\dots \\ (x_k^{n-i})_{n-p < i \leq n, 1 \leq k \leq n} \end{bmatrix}$$

and $\det(A'_\delta) = 0$ since A'_δ has a row of zeros. If $n-1 < p$, then $A'_\delta = (x_k^{n-i})$. Hence for $n-1 < p$, $\det M' = \det A'_\delta = \det(x_k^{n-i})$, the Vandermonde determinant.

Now let $\alpha = \lambda + \delta$ and the result follows. \diamond

The determinant in (1) is called the Jacobi-Trudi determinant and the determinant in (6) can be considered to be a Jacobi-Trudi determinant analogue. It is well known in classical theory that the Jacobi-Trudi determinant has a dual version, namely (3) above. We can prove a dual version of the modular Jacobi-Trudi determinant (6) by using a proof technique from Aitken [1] (Macdonald [6, p.15]).

Theorem 3.2 *For partitions λ and μ ,*

$$s'_{\lambda/\mu}(x) = \det(e'_{\lambda'_i - \mu'_j - i + j}(x)).$$

Proof: We prove

$$\det(h'_{\lambda'_i - \mu'_j - i + j}(x)) = \det(e'_{\lambda'_i - \mu'_j - i + j}(x)).$$

Let N be a positive integer and consider the following matrices with $N+1$ rows and columns,

$$H' = (h'_{i-j}(x))_{0 \leq i, j \leq N}, \text{ and } E' = ((-1)^{i-j} e'_{i-j}(x))_{0 \leq i, j \leq N}.$$

Both H' and E' are lower triangular with 1's down the diagonal, so that $\det H' = \det E' = 1$; moreover, (5) shows that H' and E' are inverses of each other. It follows that each minor of H' is equal to the complementary cofactor of E'^T , the transpose of E .

Let λ and μ be partitions of length $\leq m$ such that λ' and μ' have length $\leq q$ where $m+q = N+1$. Consider the minor of H' with row indices $\lambda_i + m - i$ ($1 \leq i \leq m$) and column indices $\mu_i + m - i$ ($1 \leq i \leq m$). By (1.7),

p. 3 of Macdonald [6], the complementary cofactor of E'^T has row indices $m - 1 + j - \lambda'_j$ ($1 \leq j \leq q$) and column indices $m - 1 + j - \mu'_j$ ($1 \leq j \leq q$). Hence we have

$$\det(h'_{\lambda'_i - \mu'_j - i + j}(x))_{1 \leq i, j \leq m} = (-1)^{|\lambda| + |\mu|} \det((-1)^{\lambda'_i - \mu'_j - i + j} e'_{\lambda'_i - \mu'_j - i + j}(x))_{1 \leq i, j \leq q}.$$

The minus signs cancel out, and thus we have

$$\det(h'_{\lambda'_i - \mu'_j - i + j}(x))_{1 \leq i, j \leq m} = \det(e'_{\lambda'_i - \mu'_j - i + j}(x))_{1 \leq i, j \leq q} \diamond$$

The final identities we describe are a modular analogues of the Giambelli identity for Schur functions. Although the three determinants described here appear to have different forms, they are in fact manifestations of the same phenomenon, namely the planar decomposition of the diagram of the partition (see Hamel and Goulden [3]). In the first instance, the Jacobi–Trudi determinant has subscripts determined by a decomposition into rows in the diagram. In the second instance, the dual Jacobi–Trudi determinant has subscripts determined by a decomposition into the columns in the diagram. Finally, the Giambelli determinant has subscripts determined by a decomposition into the principal hooks in the diagram. The Frobenius notation is the most natural to describe hooks and we use it for Theorem 3.3.

Theorem 3.3 *For partition $\lambda = (\alpha|\beta) = (\alpha_1, \dots, \alpha_r|\beta_1, \dots, \beta_r)$,*

$$s'_\lambda(x) = s'_{(\alpha|\beta)}(x) = \det(s'_{(\alpha_i|\beta_j)}(x))_{1 \leq i, j \leq r}.$$

Proof: Let λ be any partition with less than or equal to n parts. Note that, similar to Macdonald [6, p. 30], for the partition $\lambda = (a + 1, 1^b)$,

$$s'_{(a|b)}(x) = h'_{a+1}(x)e'_b(x) - h'_{a+2}(x)e'_{b-1}(x) + \dots + (-1)^b h'_{a+b+1}(x). \quad (9)$$

This comes from from expanding the Jacobi–Trudi determinant in the definition of the modular Schur function.

Now consider the modular Jacobi–Trudi matrix, $(h'_{\lambda'_i - i + j}(x))_{1 \leq i, j \leq n}$. Multiply it on the right by the matrix $((-1)^{j-1} e'_{n+1-j-k}(x))_{1 \leq j, k \leq n}$ and use the identity above in (9) to obtain the matrix $(s'_{(\lambda'_i - i | n - k)}(x))_{1 \leq i, k \leq n} = (s'_{(\alpha_i | n - k)}(x))_{1 \leq i, k \leq n}$. This matrix will have several rows which are all zeros except for a single 1

or -1 (this comes from the definition in Section 1 of $s_{(a|b)}$ for a, b negative). The other rows contain symmetric functions whose subscripts are composed of α_i and β_j . Now by taking determinants of both sides by expanding the Schur function matrix about the mostly zero rows, and applying the result of Macdonald [6], Section 1, exercise 4, to prove that in this expansion the columns which survive are those corresponding to $\beta_j, 1 \leq j \leq r$, we obtain the result. \diamond

The three determinants discussed here are not the only determinants equal to the classical Schur function. Lascoux and Pragacz have proved two additional determinants, the skew Giambelli [4] and rim ribbon [5] determinants, are equal to $s_{\lambda/\mu}(x)$. As their techniques are entirely algebraic and rely on manipulations of matrix minors—similar in some respects to the proof of Theorem 3.2—their proofs should carry over to modular Schur functions as well, so that at least two additional determinants are equal to $s'_{\lambda/\mu}(x)$.

Furthermore, using the techniques of Hamel and Goulden [3], an entire family of determinants for the modular Schur function may be possible. One limiting factor in this respect, however, is that the proof techniques of Hamel and Goulden [3] are combinatorial, whereas there does not yet appear to be a combinatorial interpretation of the modular Schur function.

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